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The Effective Interface Potential for a Superconducting Layer

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The effective interface potential is derived for a superconducting layer attached to a wall. The expression applies to the neighborhood of a continuous wetting or delocalization transition, which exists for type I superconductors with a negative extrapolation length. From this potential a number of features can be easily derived, such as the locus of the phase transition and the critical exponents. Whereas the order parameter exponent is universal, other exponents, like the susceptibility exponent, are not.

KEY WORDS: Superconductivity; wetting; interface potential.

1. INTRODUCTION

In a type I superconductor at or near bulk coexistence between the superconducting and normal states, the properties close to the surface are strongly influenced by the material in contact with the superconductor.[1] In particular, a superconducting layer, or sheath, of microscopic and even macroscopic thickness can be induced at the surface of a superconductor which, in the bulk, is in the normal state (see refs. 2 for general references on superconductivity). In fact, it was recently shown[3] that type I superconductors exhibit a rich variety of surface phenomena when the so-called extrapolation length is negative. A negative extrapolation length has been observed in a number of materials, and can also be achieved by appropriate treatment of the boundary with which the superconductor makes contact (see ref. 4 for a review). For such systems the Ginzburg–Landau (GL)

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theory is quantitatively correct and, on the basis of the GL equations, a detailed description of the surface behavior was given in ref. 3. The analysis was based on a numerical solution of the GL equations.

On the other hand, the Landau free energy, in the guise of an effective interface potential, plays a central role in the extensive literature on wetting phenomena (see ref. 5 for a review). The effective interface potential, in this context, equals the equilibrium (Gibbs) free energy, with the additional constraint that the surface layer of the phase favored by the wall has a prescribed thickness. It is the potential experienced by a flat interface in the neighborhood of the wall. From this interface potential the properties of a continuous or weakly first order wetting transition immediately follow. In particular, an analysis exists for the case of a two-component order parameter,[6] which shows that nonuniversality of critical exponents is a natural consequence of the existence of two distinct length scales, each associated with a different component of the order parameter. The problem of a superconducting sheath induced by the wall is basically a wetting or delocalization problem. This aspect of the phenomenology described in ref. 3 can therefore conveniently be discussed in a similar framework.

The construction of an interface potential from an underlying theory (in our case, the full Ginzburg–Landau equations) is, in general, a nontrivial task.[7,8] It is the aim of this paper to derive the interface potential for our delocalization problem in a systematic expansion in the GL parameter κ, which is the ratio of the magnetic penetration depth λ to the superconducting coherence length ξ. In the limit $\kappa \to 0$, an interface potential for the superconducting/normal interface has recently been constructed by Blossey and Indekeu.[9] However, for the discussion of the continuous delocalization transition, which is our concern here, it is essential to keep κ finite, since this transition only occurs on the interval $0.374 < \kappa < 1/\sqrt{2}$.[3]

To realize this program we recall, in Section 2, the basic expression for the surface free energy of a superconducting layer in contact with a wall. Minimization of this functional with respect to the relevant fields yields Euler equations precisely analogous to the equations of motion of a classical dynamics problem in two spatial dimensions. The solution of this dynamics problem, for a given set of parameters, thus provides the exact equilibrium surface state, and integration of the corresponding order parameter profile yields the equilibrium surface free energy. However, our aim is, basically, *not* to calculate equilibrium properties, since the interface potential is the free energy with the additional *constraint* of a prescribed thickness of the superconducting layer. Nevertheless, close to the delocalization transition, this constraint is sufficiently mild that the constrained dynamical trajectories are, in the relevant region, close to those corresponding to unconstrained equilibrium. Thus, in order to construct the

interface potential, we need the *equilibrium* wall–superconducting interface for parameter values at and near the delocalization transition. As a consequence we construct in Section 3 the equilibrium solution for a wall–superconducting interface at criticality, using κ as an expansion parameter. In Section 4 we extend this solution to the neighborhood of the delocalization transition. Expansion in κ turns out to be very effective here. The final step is made in Section 5, where we introduce the constraint of a given thickness of the superconducting layer, and calculate the corresponding interface potential. For this calculation the equilibrium solutions of the previous sections are essential, and the parameters of those solutions are shown to be linked in a transparent way to those of the delocalization transition. The paper closes with a discussion of the relevant physics which can be extracted from the interface potential.

2. THE GINZBURG–LANDAU SURFACE FREE ENERGY

In the GL theory the state of the superconductor is characterized by the vector potential $\mathbf{A}(\mathbf{r})$ and the complex order parameter $\psi(\mathbf{r})$[2]. In the simple geometry of a flat wall with a magnetic field parallel to the wall, we may use for $\mathbf{A}(\mathbf{r})$ the gauge

$$\mathbf{A}(\mathbf{r}) = (0, A(x), 0) \tag{1}$$

and a real function $\psi(x)$ depending only on the coordinate x perpendicular to the wall. Then the boundary condition for $\psi(x)$ at the wall $x = 0$ reads

$$d\psi(x)/dx|_{x=0} = \psi(0)/b \tag{2}$$

where b is a material constant, the so-called extrapolation length. The corresponding boundary condition for $A(x)$ relates to the magnetic field H and reads

$$dA(x)/dx|_{x=0} = \mu_0 H \tag{3}$$

Several options exist to scale the physical quantities such as to make them dimensionless. It is convenient here to scale ψ with its bulk value ψ_b

$$\varphi(x) = \psi(x)/\psi_b \tag{4}$$

such that φ approaches 1 in the fully developed superconducting state. The spatial coordinate is measured in terms of the magnetic penetration depth λ. With this scaling one can derive the GL equations and the boundary

conditions (2) and (3) by minimizing the dimensionless free energy functional,[3]

$$\sigma[\varphi, a] = \int_0^\infty dx[(\dot{\varphi}/\kappa)^2 - \varphi^2 + \varphi^4/2 + a^2\varphi^2 + (\dot{a} - h)^2] + \tau\varphi^2(0)/\kappa^2 \tag{5}$$

Here a is the appropriately scaled version of A, and h likewise that of H, with $\dot{a} \equiv da/dx$. The parameter τ is related to b by (note that our τ is κ times the τ used in ref. 3)

$$\tau = \lambda/b \tag{6}$$

Upon variation of σ with φ and a we obtain the GL equations

$$\ddot{\varphi}/\kappa^2 = -\varphi + a^2\varphi + \varphi^3 \tag{7}$$

$$\ddot{a} = a\varphi^2 \tag{8}$$

and variation of σ with respect to $\varphi(0)$ and $a(0)$ gives the boundary conditions at the wall,

$$\dot{a}(0) = h, \qquad \dot{\varphi}(0) = \tau\varphi(0) \tag{9}$$

There are two additional conditions, the form of which depends on the physical state far from the wall, i.e., in the bulk. Equations (7) and (8), with x playing the role of "time," show that our problem is equivalent to one in classical dynamics, in which a point particle with anisotropic mass[3] moves in two dimensions (a, φ) in the potential

$$V(a, \varphi) = -\tfrac{1}{2}a^2\varphi^2 - \tfrac{1}{4}(\varphi^2 - 1)^2 \tag{10}$$

Clearly, this potential has a maximum at $a = 0$, $\varphi = 1$, where $V(0, 1) = 0$. In the dynamical analog, the boundary conditions at the wall play the role of initial conditions, whereas those in the bulk are final ones. The physics associated with delocalization requires, in general, bulk coexistence between two phases. In our problem, these two phases are the superconducting and normal states. We limit ourselves to strict coexistence here, which implies for h the value $h_c = 1/\sqrt{2}$. If the state is normal as $x \to \infty$, the final conditions are $\dot{a}(\infty) = 1/\sqrt{2}$, $\varphi(\infty) = 0$. If the bulk state is superconducting, the final conditions become $a(\infty) = 0$, $\varphi(\infty) = 1$. In the latter case the total energy in the dynamical analog is, clearly, $\mathscr{E} = 0$. At coexistence the total energy must also vanish when the asymptotic state is normal.

[3] The scaling used in ref. 3 removes the anisotropy in the dynamical analog. However, the scaling used here is more convenient for our present purposes.

The problem contains only two parameters: the ratio κ and the wall parameter τ. As κ and b are material constants we may, with b *negative*, think about τ as a temperature parameter running from small *negative* values at low temperatures to large *negative* values for temperatures close to the bulk transition point (at zero field).

3. THE CRITICAL WALL–SUPERCONDUCTOR INTERFACE

We shall now construct a solution of Eqs. (7) and (8) which describes the wall–superconducting interface. We must then supplement the initial conditions (9) by the final conditions $a(\infty)=0$, $\varphi(\infty)=1$. For given parameters κ and τ this completely specifies the problem. In this section we do not consider arbitrary τ, however, but specialize to τ_c, corresponding to the critical temperature for a continuous "wetting" or delocalization transition. That the solution found in this section has this status will only become clear in Section 5. The solution will be constructed as a power series in κ.

We write φ as a perturbation of the bulk value

$$\varphi = 1 + f \tag{11}$$

and consider f and a as small. Then (7) and (8) become

$$\ddot{f} - 2\kappa^2 f = \kappa^2[a^2(1+f) + 3f^2 + f^3] \tag{12}$$

$$\ddot{a} - a = 2af + af^2 \tag{13}$$

We solve (12) and (13) in a leapfrog scheme: in first order we start with a, then determine f in second order, a in third order, etc. Thus we determine a_1 from

$$\ddot{a}_1(x) - a_1(x) = 0 \tag{14}$$

of which we take the dying exponent

$$a_1(x) = Ae^{-x} \tag{15}$$

since $a=0$ in the superconducting phase (with our choice of gauge). We then linearize (12) to get

$$\ddot{f}_2(x) - 2\kappa^2 f_2(x) = \kappa^2 a_1^2 = \kappa^2 A^2 e^{-2x} \tag{16}$$

In solving (16) we could freely add a solution of the homogeneous equation. However, we insist that the homogeneous contribution to f_2 vanishes

here. As will become apparent in Section 5, this is equivalent to setting $\tau = \tau_c$. Our solution of (16) is then

$$f_2(x) = \frac{\kappa^2}{4 - 2\kappa^2} A^2 e^{-2x} \tag{17}$$

The procedure now becomes clear. In third order we determine a_3 from (13), with the terms containing a factor A^3 on the right-hand side,

$$\ddot{a}_3(x) - a_3(x) = 2a_1 f_2 = \frac{\kappa^2}{2 - \kappa^2} A^3 e^{-3x} \tag{18}$$

The solution is

$$a_3(x) = \frac{\kappa^2}{8(2 - \kappa^2)} A^3 e^{-3x} \tag{19}$$

In this way one continues. The full solution can be summarized as

$$f(x) = \sum_{j=1} f_{2j} A^{2j} e^{-2jx} \tag{20}$$

$$a(x) = \sum_{j=0} a_{2j+1} A^{2j+1} e^{-(2j+1)x} \tag{21}$$

in which the coefficients a_{2j+1} and f_{2j}, to be determined recursively, depend on κ only. We list the first few explicitly,

$$a_1 = 1 \tag{22}$$

$$f_2 = \frac{\kappa^2}{4 - 2\kappa^2} a_1 \tag{23}$$

$$a_3 = \frac{1}{4} a_1 f_2 \tag{24}$$

$$f_4 = \frac{\kappa^2}{16 - 2\kappa^2} (2a_1 a_3 + a_1^2 f_2 + 3f_2^2) \tag{25}$$

$$a_5 = \frac{1}{24} (2a_1 f_4 + 2a_3 f_2 + a_1 f_2^2) \tag{26}$$

The important point is that in each step involving f_{2j}, a factor κ^2 is added such that for small κ the series rapidly converges. Convergence is accelerated by the increasingly large factors appearing in the denominators. Thus a_1 is of order κ^0, f_2 and a_3 are of order κ^2, f_4 and a_5 are of order κ^4, etc.

The series (20) and (21) represent a full solution to the nonlinear problem (12) and (13). We still have to consider the initial conditions (9), and we have only one parameter A to play with. We use A to match the first condition, which reads at coexistence

$$\dot{a}(0) = 1/\sqrt{2} \tag{27}$$

With (21) we may write this as

$$\sum_{j=0} a_{2j+1} A^{2j+1}(2j+1) = -1/\sqrt{2} \tag{28}$$

As the a_{2j+1} contain increasing powers of κ^2, one can solve for A as a power series in κ^2. The result is

$$A_c = \frac{1}{\sqrt{2}}\left(-1 + \frac{3\kappa^2}{32} + \frac{57\kappa^4}{2048} + \frac{1879\kappa^6}{196608} + \cdots\right) \tag{29}$$

We may now ask for which value of τ the solution (21), with $A = A_c$ given by (29), matches the second initial condition of (9). We call this value τ_c, anticipating that it will be the critical τ for continuous delocalization of the interface (see Section 5). Equation (9) gives

$$\tau_c = \frac{\dot{f}_c(0)}{1 + f_c(0)} \tag{30}$$

with $f_c(x)$ the value of $f(x)$ for $A = A_c$. From the forgoing formulas we can derive an expansion for τ_c in powers of κ^2,

$$\tau_c = -\frac{\kappa^2}{4}\left(1 + \frac{9\kappa^2}{32} + \frac{27\kappa^4}{256} + \cdots\right) \tag{31}$$

This value may be compared with the numerically obtained locus $\tau_c(\kappa)$[3]. Expression (31) fits already the least favorable case, with κ taking its maximal value $\kappa = 1/\sqrt{2}$, within 1%. (Note again that our τ is κ times the τ of ref. 3.)

4. THE WALL–SUPERCONDUCTOR INTERFACE NEAR τ_c

As a first generalization of the results of Section 3 we now consider the wall–superconducting interface [again with final conditions $a(\infty) = 0$,

$\varphi(\infty)=1$] at $\tau=\tau_c+\Delta\tau$, with $\Delta\tau$ sufficiently small that linearization in the deviation is meaningful. The solution (20) and (21) will now be of the form

$$f(x)=f_c(x)+\Delta B\, g(x)+\Delta C\, h(x) \tag{32}$$

$$a(x)=a_c(x)+\Delta B\, b(x)+\Delta C\, c(x) \tag{33}$$

where $f_c(x)$ and $a_c(x)$ are the functions (20) and (21) with A replaced by A_c. The functions $g(x)$ and $b(x)$ result from a shift from A_c to

$$A=A_c(1+\Delta B) \tag{34}$$

such that $g(x)$ and $b(x)$ are generated as in (20) and (21) with coefficients g_{2j} and b_{2j+1} given by

$$g_{2j}=(2j)f_{2j}, \qquad b_{2j+1}=(2j+1)\,a_{2j+1} \tag{35}$$

(and $A=A_c$). The new functions $h(x), c(x)$ are results of adding a homogeneous solution of Eq. (16) to f_2.

The coefficients ΔB and ΔC are assumed to be of linear order in $\Delta\tau$. That is, we linearize (12) and (13) around $f_c(x)$ and $a_c(x)$ to obtain

$$\ddot{h}-2\kappa^2 h=\kappa^2(Fh+Hc) \tag{36}$$

$$\ddot{c}-c=Hh+Gc \tag{37}$$

where the functions F, G, and H can be expressed in terms of f_c and a_c as

$$F=6f_c+3f_c^2+a_c^2 \tag{38}$$

$$G=2f_c+f_c^2 \tag{39}$$

$$H=2a_c+2a_c f_c \tag{40}$$

These functions can all be written in terms of a series, e.g.,

$$F(x)=\sum_{j=1} F_{2j}(x) \tag{41}$$

with

$$F_{2j}(x)=F_{2j}A_c^{2j}e^{-2jx} \tag{42}$$

Likewise, G is a series in even powers of e^{-x}, while the series for H contains odd powers only.

The perturbation scheme for the Eqs. (36) and (37) is initiated by the term

$$h_0(x) = e^{-\kappa\sqrt{2}\,x} \tag{43}$$

which generates the series

$$h(x) = \sum_{j=0} h_{2j} A_c^{2j} \exp[-(2j+\kappa\sqrt{2})\,x] \tag{44}$$

$$c(x) = \sum_{j=0} c_{2j+1} A_c^{2j+1} \exp[-(2j+1+\kappa\sqrt{2})\,x] \tag{45}$$

where the coefficients are determined recursively as

$$h_0 = 1 \tag{46}$$

$$c_1 = \frac{H_1 h_0}{(1+\kappa\sqrt{2})^2 - 1} = \frac{1}{\kappa\sqrt{2}\,(1+\kappa/\sqrt{2})} \tag{47}$$

$$h_2 = \frac{\kappa^2(F_2 h_0 + H_1 c_1)}{(2+\kappa\sqrt{2})^2 - 2\kappa^2} = \frac{\kappa^2}{4+4\kappa\sqrt{2}}(2c_1 + 1 + 6f_2) \tag{48}$$

Note that h_0 is of order κ^0, c_1 of order κ^{-1}, h_2 and c_3 of order κ, h_4 and c_5 of order κ^3, etc.

With ΔB and ΔC at our disposal we can match the initial conditions (9). The first one gives

$$\Delta B\,\dot{b}(0) + \Delta C\,\dot{c}(0) = 0 \tag{49}$$

by which we can express ΔB in terms of ΔC. Expanding in powers of κ, we find

$$\Delta B = -\frac{\Delta C}{\kappa\sqrt{2}}\left(1 + \frac{\kappa}{\sqrt{2}} - \frac{\kappa^2}{2} - \frac{\kappa^3}{4\sqrt{2}} + \cdots\right) \tag{50}$$

The other initial condition provides a relation between $\Delta\tau$, ΔB, and ΔC,

$$\tau_c + \Delta\tau = \frac{\dot{f}_c(0) + \Delta B\,\dot{g}(0) + \Delta C\,\dot{h}(0)}{1 + f_c(0) + \Delta B\,g(0) + \Delta C\,h(0)} \tag{51}$$

In view of (30) the linearized form of (51) is

$$\frac{\Delta\tau}{\tau_c} = \Delta B\left(\frac{\dot{g}(0)}{\dot{f}_c(0)} - \frac{g(0)}{1+f_c(0)}\right) + \Delta C\left(\frac{\dot{h}(0)}{\dot{f}_c(0)} - \frac{h(0)}{1+f_c(0)}\right) \tag{52}$$

As all these quantities are known in expansions in κ we find, using (50),

$$\Delta C = -\Delta\tau\left(\frac{1}{\kappa\sqrt{2}}+\frac{3}{8}+\frac{\kappa}{32\sqrt{2}}+\frac{13\kappa^2}{128}-\frac{129\kappa^3}{1024\sqrt{2}}+\frac{135\kappa^4}{1024}+\cdots\right) \tag{53}$$

Thus (32) and (33) represent a solution of the wall–superconductor interface when initial conditions in the form (50) and (53) are used to express ΔB and ΔC in terms of $\Delta\tau$, in the linear regime around τ_c.

5. THE EFFECTIVE INTERFACE POTENTIAL

The solution constructed in Section 4 represents a wall–superconductor interface, since the asymptotic behavior for $x \to \infty$ corresponds to the superconducting state in which $\varphi = 1$ and the magnetic field $\dot{a} = 0$. In the dynamical language, the trajectory climbs to the top of the potential landscape at $\varphi = 1$ and $a = 0$. The wall–superconductor interface is part of the total surface profile for the case when the superconducting–normal interface is *delocalized*, i.e., when the superconducting sheath is of essentially infinite (macroscopic) thickness. The remaining part of the surface profile is then a *free* superconducting–normal interface.

However, for temperatures below that of the delocalization transition, the free energy is lowered by a superconducting layer of *finite* thickness near the wall, in coexistence with the normal state in the bulk. Dynamically this means that the trajectory bends away from the top and runs toward $\varphi = 0$ and $a = \infty$, obeying the final conditions $\varphi(\infty) = 0$ and $\dot{a}(\infty) = 1/\sqrt{2}$. Thus we are forced to include growing exponentials in the dynamical solution. This presents a problem, because the growing expontials diverge for large values of x and cannot be handled in the perturbation scheme, nor in the integrals. In ref. 6 the way to solve this problem was outlined.

One has to consider the trajectory of a free interface between a superconducting and a normal state. It runs from the top $\varphi = 1, a = 0$ for $x \to -\infty$ to $\varphi = 0$, $a = \infty$ for $x \to \infty$. The location of the interface is a free parameter, which does not influence the form of the trajectory. We take it at $x = l$ (as $x = 0$ is already reserved for the position of the wall). Such a free interface is, for $x \ll l$, well described by the functions

$$f_i(x) = -Q\exp[\kappa\sqrt{2}\,(x-l)], \qquad a_i(x) = \exp[x-l] \tag{54}$$

where Q is a characteristic of the interface, defined as

$$Q = \lim_{x\to-\infty} f_i(x)/[a_i(x)]^{\kappa\sqrt{2}} \tag{55}$$

Q depends only on κ and its calculation poses a separate problem. We give here the first few terms in an expansion in powers of κ, which we have computed using a technique developed by Boulter and Indekeu[10]:

$$Q = \tfrac{1}{2}\{1 + [\kappa/2(\sqrt{2})] \ln(2\kappa/K_0^4) + \cdots\} \tag{56}$$

K_0 is a numerical constant which to zeroth order in κ reads $K_0 = 0.795$.

The interface potential is the surface free energy with the sheath thickness constrained to take the value l. In order to construct the interface potential, we must first consider the dynamical solutions corresponding to sheaths with finite thickness l. They have the form

$$f(x) = f_0(x) + \delta f(x) \tag{57}$$

$$a(x) = a_0(x) + \delta a(x) \tag{58}$$

where $f_0(x)$ and $a_0(x)$ are the solutions (32) and (33) for a given $\Delta\tau$. The small increments will be represented as

$$\delta f(x) = \delta B\, g(x) + \delta C\, h(x) + \delta D\, j(x) + \delta E\, k(x) \tag{59}$$

$$\delta a(x) = \delta B\, b(x) + \delta C\, c(x) + \delta D\, d(x) + \delta E\, e(x) \tag{60}$$

The new set $j(x), d(x)$ is very similar to $h(x), c(x)$. It is generated by the seed

$$j_0(x) = e^{\kappa\sqrt{2}\,x} \tag{61}$$

and leads to the series

$$j(x) = \sum_{j=0} j_{2j} A_c^{2j} \exp[-(2j - \kappa\sqrt{2})\, x] \tag{62}$$

$$d(x) = \sum_{j=0} d_{2j+1} A_c^{2j+1} \exp[-(2j+1-\kappa\sqrt{2})\, x] \tag{63}$$

In fact, one can obtain $j(x)$ from $h(x)$ and $d(x)$ from $c(x)$ by changing the sign of κ.

The construction of the set $k(x), e(x)$ is slightly more involved. The seed for the recursion is

$$e_{-1}(x) = e^x/A_c \tag{64}$$

The next quantity is $k_0(x)$. It follows from the equivalent of (36),

$$\ddot{k}_0(x) - 2\kappa^2 k_0(x) = \kappa^2 H_1(x)\, e_{-1}(x) = 2\kappa^2 \tag{65}$$

with the solution

$$k_0(x) = -1 \tag{66}$$

In the following step a complication arises. The equation for $e_1(x)$ reads, in view of (37),

$$\ddot{e}_1(x) - e_1(x) = H_1(x)\, k_0(x) + G_2(x)\, e_{-1}(x) = 2(f_2 - 1)\, e^{-x} \tag{67}$$

This has the solution

$$e_1(x) = (1 - f_2)\, xe^{-x} \tag{68}$$

From here on we get terms with and without a factor x. The resulting series is

$$k(x) = \sum_{j=0} (k_{2j} + k'_{2j} x)\, A_c^{2j} \exp[-2jx] \tag{69}$$

$$e(x) = \sum_{j=-1} (e_{2j+1} + e'_{2j+1} x)\, A_c^{2j+1} \exp[-(2j+1)\, x] \tag{70}$$

The coefficients can again be determined recursively, as we already started doing in (64)–(68). For the leading terrain the interface potential we do not need coefficients beyond those given here.

With (57)–(60) we have a solution with four free amplitudes, provided they all are small. The equilibrium solution has to satisfy four conditions, two initial conditions and two final ones. In other words, for the equilibrium solution the number of equations matches the member of unknowns. However, we are *not* aiming directly for the equilibrium solution and its surface free energy, but want to construct the interface potential for a prescribed thickness l of the superconducting sheath. What we need to compute is then the free energy *differences* corresponding to interfaces with different values of l. The equilibrium thickness l_{eq} follows from minimization of this interface potential. Clearly, these free energy differences must be finite. With growing exponentials allowed in our solutions, there seems to be a serious problem. As shown in ref. 6, however, the corresponding divergence problems can be circumvented by subtraction from the free energy of the profile considered, that of the free interface located at l. With the assumption that the equilibrium value l_{eq} is large and that the given

value l is not too different from l_{eq}, we can then implement the constraint by insisting that the trajectory asymptotically matches that of a free interface located at l. From (54) and the leading terms in (59) and (60) one can check that this is accomplished by setting

$$\delta D = -Qe^{-\kappa\sqrt{2}\,l}, \qquad \delta E = A_c e^{-l} \tag{71}$$

Taking l sufficiently large makes the δD and δE small enough to justify the linearization of the GL equations. The stipulation (71) removes divergences associated with growing exponentials. An additional source of divergences in the free energy exists, however. If two constrained solutions would correspond to different *dynamical* energies, this would lead to a *linear* divergence in the free energy difference. It is consequently essential in the construction of the interface potential to insist that all constrained solutions have the same dynamical energy. From (5) and (10) it follows that twice the dynamical energy reads

$$2\mathscr{E} = (\dot{\varphi}/\kappa)^2 + \dot{a}^2 - a^2\varphi^2 - \tfrac{1}{2}(\varphi^2 - 1)^2 \tag{72}$$

At coexistence, the common value for all acceptable solutions is $\mathscr{E} = 0$. When we evaluate $\mathscr{E}$ from (57)–(60), we have to multiply series with many exponentials. We can use the fact that $\mathscr{E} = 0$ to restrict the calculation to products that lead to a constant. All other combinations automatically cancel. To second order in the δ's the energy requirement reads

$$\begin{aligned}\mathscr{E} = &-4(\Delta C + \delta C)\,\delta D - 2(1 + \Delta B + \delta B)\,\delta E \\ &-2(1 + \Delta B - 3\Delta C)\,\delta D\,\delta E + (1 + 2\Delta B)(\delta E)^2 = 0\end{aligned} \tag{73}$$

Note that for $\delta D = \delta E = 0$ (as in Section 4), $\mathscr{E}$ vanishes automatically.

With the stipulation (71) we now have three equations, the energy requirement (73) and the two initial conditions (9), to determine the remaining constants δB and δC. In general, no choice of δB and δC can fulfill all requirements. Since the energy requirement is unexceptionable, we must be willing, when optimizing the interface potential, to relax somewhat the initial conditions. Note that there is no paradox here: the constrained solutions basic to the interface potential are *not* solutions to the full dynamical problem, except when $l = l_{eq}$.

Thus we must find the best values of δB and δC by optimizing the free energy under the constraint of a given $\mathscr{E}$, which at coexistence takes the value $\mathscr{E} = 0$. We shall see that, to leading order in the interface potential, we do not have to carry out this program in detail.

We now compare the free energy of the sheath solutions with that of a wall–superconductor interface plus a free interface. The latter two are given by

$$\sigma_0 = \int_0^\infty dx\, \gamma_0(x) + \tau\varphi_0^2(0) \tag{74}$$

$$\sigma_i = \int_{-\infty}^{\infty} dx\, \gamma_i(x) \tag{75}$$

where $\gamma(x)$ is the integrand of (5). The difference of σ for the sheath and $\sigma_0 + \sigma_i$ gives the interface potential $\delta\sigma$. We write it as

$$\delta\sigma = \int_0^\infty dx[\gamma(x) - \gamma_i(x)] - \int_{-\infty}^0 dx\, \gamma_i(x) + \tau\varphi^2(0) - \sigma_0 \tag{76}$$

In the first integral the diverging exponentials cancel by the choice (71). In the second integral we may use (54) in the whole interval. Thus (76) is a well-defined expression for the interface potential.

By working out the separate terms we distinguish three classes: those involving f_0 and a_0 only, those linear in δf and δa, and the higher order terms in the δ's. The terms in the first class cancel exactly due to the subtraction of σ_0. The linear terms mostly vanish due to the variational character of the expression (5) for σ, and the fact that f_0 and a_0 satisfy the equations of motion *and* the boundary conditions at $x = 0$. In linear order all that remains is a term from the boundary at $x \to \infty$, resulting from partial integration of the terms with $\delta\dot{f}$ and $\delta\dot{a}$. As a result we find

$$\delta\sigma = \lim_{x \to \infty} [2\dot{f}_0(x)\, \delta f(x)/\kappa^2 + 2\dot{a}_0(x)\, \delta a(x)] + O(\delta^2) \tag{77}$$

From the series for the various functions it is easy to see which terms survive as $x \to \infty$. We find

$$\delta\sigma = -\frac{2\sqrt{2}}{\kappa} \Delta C\, \delta D - 2(1 + \Delta B)\, \delta E + O(\delta^2) \tag{78}$$

Note that the second integral in (76) leads to contributions of order δ^2 only. Thus we have found the leading order of the interface potential. Using (71), one finds the explicit dependence on l as

$$\delta\sigma(l) = \frac{2\sqrt{2}\, Q}{\kappa} \Delta C\, e^{-\kappa\sqrt{2}\, l} - 2A_c(1 + \Delta B)\, e^{-l} \tag{79}$$

Equation (79) is our final result for the interface potential. The leading terms in a κ expansion of Q are given in (56), ΔC is linearly related to $\Delta\tau$ by (53), and A_c, which is negative, is given by (29).

Since the coefficient of the second exponential, e^{-l}, is positive, minimization of $\delta\sigma$ shows that, for negative ΔC (low temperatures), l_{eq} is finite. On the other hand, for $\Delta C > 0$ (higher temperatures), the equilibrium thickness of the superconducting sheath diverges (i.e., becomes macroscopic). In other words, there is a delocalization transition at $\tau = \tau_c$, as anticipated in Section 3

6. CONCLUDING REMARKS

In this paper we have constructed the interface potential for the superconducting–normal interface close to the delocalization transition. The fact that such a potential can be systematically constructed on the basis of an underlying theory is in itself nontrivial. The standard difficulty of finding an effective means of constraining the interface could be solved here, as in the technically simpler case studied in ref. 6, by stipulating the value of small amplitudes of growing exponentials close to the (superconducting) top in the dynamical landscape.

Note that our procedure differs from the prescription of Jin and Fisher[7] in the following way. They insist that the boundary conditions at the wall are strictly obeyed. The boundary conditions at the wall and at infinity, together with the imposition of the thickness of the layer, do not permit the interface profile to assume everywhere its optimal shape. Jin and Fisher compromise by gluing together two optimal profiles at the position of the interface, which leads to a (small) singularity in the profile at the position of the interface. Our profile obeys everywhere-the minimalization equations (7)–(8), and has therefore no singularities near the position of the interface. Instead, we relax the boundary conditions at the wall in order to have the freedom to prescribe the thickness of the interface. The free energy associated with deviations from the boundary conditions turns out to be of higher order than that due to the fluctuations in the position of the the interface. Therefore, the fluctuations at the wall do not contribute to the leading order of the interface potential.

Minimization of the interface potential (79) immediately gives

$$l_{eq} = \frac{\ln[A_c(1+\Delta B)/(2Q\,\Delta C)]}{1-\kappa\sqrt{2}} \tag{80}$$

which shows that l_{eq} diverges logarithmically as $\Delta C \sim -\Delta\tau \to 0^-$. The logarithmic divergence is in agreement with the numerical solution obtained

in ref. 3. The nature of this divergence of the effective order parameter l_{eq} is clearly universal. Note that the amplitude diverges when κ approaches $1/\sqrt{2}$, which was already noted by Speth[11] (see ref. 5 for a review of Speth's work). This was interpreted to imply that only at $\kappa = 1/\sqrt{2}$ does a wetting transition take place. It should be clear that we do not agree with that interpretation.

On the other hand, it follows from (79) and (80) that the equilibrium surface free energy is singular as the delocalization transition is approached from below, with the singular contribution behaving as

$$\delta\sigma_{eq} \sim -(-\Delta C)^{1/(1-\kappa\sqrt{2})} \sim -(\Delta\tau)^{1/(1-\kappa\sqrt{2})} \tag{81}$$

demonstrating that most other exponents depend on the material parameter κ in a continuous manner. The exponent in (81) is commonly referred to as $2-\alpha_s$, which here equals γ_s, since $\beta_s = 0$, in view of the logarithmic divergence of (80). Clearly, $\alpha_s \leq 0$, starting from $\alpha_s = 0$ at $\kappa = (2\sqrt{2})^{-1}$, and with $\alpha_s \to -\infty$ as $\kappa \to (1/\sqrt{2})$.

From ref. 6 one would expect that there is a crossover from a continuous to a first-order delocalization transition when twice the exponent in the slowest exponential equals the exponent in the next to slowest one, i.e., when $\kappa = (2\sqrt{2})^{-1} = 0.354$. However, in the present case this transition is preempted by another first-order transition, that of nucleation of a macroscopic layer directly from the normal state (at the value $\kappa = 0.374$, which is rather close). This nucleation line continues[3] via a tricritical point as a line of continuous transitions to positive values of τ. An analytic theory of this transition based on the GL equations, beyond the $\kappa \to 0$ results given in ref. 9, remains an open problem.

It is remarkable how efficient the expansion in powers of κ is for this surface problem. As an example, in Section 3 we calculated the locus of the critical delocalization temperature τ_c to $\mathcal{O}(\kappa^6)$, which in the original physical variables reads

$$\frac{\lambda(T_c)}{-b} = \frac{\kappa^2}{4}\left(1 + \frac{9\kappa^2}{32} + \frac{27\kappa^4}{256} + \cdots\right) \tag{82}$$

in excellent agreement with the numerical results of ref. 3. It is our belief that the entire phase diagram given in ref. 3 can be calculated in expansions in κ, but this is beyond the scope of the present paper.

Finally, the challenge to the experimentalists is to confirm the phase diagram predicted in ref. 3. This paper adds an even more demanding challenge: Determine experimentally some nonuniversal exponent associated with the delocalization transition!

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